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SOME SETS OBEYING HARMONIC SYNTHESIS

BY

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ABSTRACT

Let X be a (not necessarily closed) subspace of the dual space B^* of a separable Banach space B. Let X_1 denote the set of all weak* limits of sequences in X. Define X_a, for every ordinal number a, by the inductive rule: $X_a = (\bigcup_{b<a} X_b)_1$. There is always a countable ordinal a such that X_a is the weak* closure of X; the first such a is called the *order* of X in B^*.

Let E be a closed subset of a locally compact abelian group. Let $PM(E)$ be the set of pseudomeasures, and $M(E)$ the set of measures, whose supports are contained in E. The set E obeys synthesis if and only if $M(E)$ is weak* dense in $PM(E)$. Varopoulos constructed an example in which the order of $M(E)$ is 2. The authors construct, for every countable ordinal a, a set E in R that obeys synthesis, and such that the order of $M(E)$ in $PM(E)$ is a.

Let G and Γ be locally compact abelian groups, each the dual of the other. Let $A = A(\Gamma)$ be the Fourier representation of the convolution algebra $L^1(G)$. The Banach space dual of A is the space $PM = PM(\Gamma)$ consisting of all the distributions S on Γ such that $\hat{S} \in L^\infty(G)$. These distributions are called *pseudomeasures*, and the dual space norm $\|S\|_{PM}$ equals $\|\hat{S}\|_\infty$. For a closed set $E \subset \Gamma$, let $PM(E)$ be the set of pseudomeasures, and $M(E)$ the set of measures, whose supports are contained in E. Let $N(E)$ be the weak* closure of $M(E)$ in PM. The largest closed ideal in A whose hull is E is

$$I(E) = \{f \in A : f^{-1}(0) \supset E\} = M(E)^\perp = N(E)^\perp,$$

and the smallest such ideal is $I_0(E)$, the closure of

$$\{f \in A : f^{-1}(0) \text{ is a neighborhood of } E\}.$$

The set E is said to *obey harmonic synthesis*, or to be a *set of synthesis*, if

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$I(E) = I_0(E)$ or, equivalently, if $N(E) = PM(E)$. For discussions of the basic theory of harmonic synthesis, see the books by Kahane ([1]) and Rudin ([5]).

When B is a separable Banach space, and X is a linear submanifold of the dual space B^*, let X_1 denote the set of all weak* limits of sequences in X. Then X_1 is also a linear submanifold. Define X_s for every ordinal number s by the inductive rule, $X_s = (\bigcup_{t<s} X_t)_1$. The first countable ordinal s such that X_s is the weak* closure of X is called the *order* of X. For constructions of linear submanifolds of every possible order in the Banach space H^∞, see [6]; in the space l^1, see [4].

Varopoulos constructed a set E such that the order of $M(E)$ is 2. In [2] we gave a version of his construction and raised the question of whether the order of $M(E)$ can be higher than 2. The purpose of this paper is to answer this question by proving the following result,

THEOREM. *For every countable ordinal number s, there is a subset E of the circle group such that E obeys harmonic synthesis and the order of M(E) is s.*

We shall consider the circle group T as identified with the real numbers modulo 2π, or with the interval $[0, 2\pi)$. We shall make use of the following standard lemma.

LEMMA. *Let F be a perfect null subset of T. For each $n \geqq 1$, let $\{I_{nk} : 1 \leqq k \leqq 2^n\}$ be a set of disjoint open intervals chosen so that $F \subset \bigcup_{k=1}^{2^n} I_{nk}$, $I_{(n+1)(2k-1)} \cup I_{(n+1)(2k)} \subset I_{nk}$, and $\max_k \operatorname{diam} I_{nk} \underset{n}{\to} 0$. For $V \in PM(F)$, let V_{nk} denote the restriction of V to $I_{nk} \cap F$.*

(1) *If there is a constant B such that*

$$\sum_{k=1}^{2^n} \| V_{nk} \|_{PM} \leqq B \text{ for every } n \geqq 1,$$

then $V \in M(F)$.

(2) *If $V \notin N(F)$, then for every $K > 0$, it is the case for all sufficiently large n that*

$$\sum_{k=1}^{2^n} \| (V - U)_{nk} \|_{PM} \geqq K \text{ for every } U \in N(F).$$

PROOF. (1) If $f \in C(T)$, and if for some n, f is constant on a neighborhood of $I_{nk} \cap F$ for each k, then

$$|\langle f, V \rangle| = \left| \sum_k \langle f, V_{nk} \rangle \right|$$
$$\leqq \sum_k |f(F \cap I_{nk})| \, \| V_{nk} \|_{PM} \leqq \| f \|_{C(T)} B.$$

Thus V defines a bounded linear functional on a dense subspace of $C(T)$, so that it must be a measure.

(2) Suppose, to the contrary, that for some K and infinitely many n there exists $U^{(n)} \in N(F)$ such that

$$\sum_{k=1}^{2^n} \|(V - U^{(n)})_{nk}\|_{PM} < K.$$

The pseudomeasures $U^{(n)}$ form a bounded sequence in $N(F)$, and we may suppose that it converges weak* to some $U \in N(F)$. It follows that $\Sigma_{k=1}^{2^n}\|(V-U)_{nk}\|_{PM} \leqq K$ for every n, so that by part (1), $V - U$ is a measure and hence $V \in N(F)$, which is false. The lemma is proved.

PROOF OF THE THEOREM. We shall proceed by induction on s to construct the sets E of the theorem, not in T, but (for $s > 1$) in the countably infinite product group T^∞, because this makes the procedure easier to describe and to visualize. Then we shall explain how to prove the theorem as stated, with $E \subset T$.

Let F be a perfect compact null set that contains no rational multiples of π, that is contained in $(1,2)$, and that disobeys synthesis. Let S be an element of $PM(F)$ that lies outside $N(F)$. The set F and the pseudomeasure S are now fixed for the duration of the proof.

Let $H = \bigcup_{q=1}^{\infty} C_q$, where

$$C_q = \{x = 2\pi p/q : p \textit{ is an integer}, x \in (1,2), \textit{and } \operatorname{dist}(x, F) \leqq 2\pi/q\}.$$

It is well known (see [1, sec. V.3]) that the set $F \cup H$ obeys synthesis in a particularly nice way. To wit, if $U \in PM(F \cup H)$, then there is a canonically defined sequence of discrete measures $\mu_q \in M(C_q)$ such that $\|\mu_q\|_{PM} \leqq \|U\|_{PM}$ and $\mu_q \to U$ weak*. Thus, of course, the set $F \cup H$ satisfies the theorem in the case $s = 1$.

Let $\{I_{nk}\}$ be a collection of open intervals as described in the lemma. Let

$$\{e_n\}_{n=0}^{\infty} \cup \{e_{nkj} : 1 \leqq n < \infty, 1 \leqq k \leqq 2^n, 1 \leqq j < \infty\}$$

be an orthogonal set of unit vectors in T^∞. Define some subsets of T^∞ as follows:

$$F_0 = \{xe_0 : x \in F\},$$

$$F_n = \{x(e_0 + 2^{-n}e_n) : x \in F\} \text{ for } n \geqq 1,$$
$$F_{nkj} = \{x(e_0 + 2^{-n}e_n + 2^{-n-k-j}e_{nkj}) : x \in (F \cup H) \cap I_{nk}\},$$
$$H_n = \bigcup_{k=1}^{2n} \bigcup_{j=1}^{\infty} F_{nkj}, \quad E = F_0 \cup \bigcup_{n=1}^{\infty} (F_n \cup H_n).$$

We claim that E obeys synthesis and that the order of $M(E)$ is 2.

Let us offer a few remarks to explain the basic idea of the construction. Let $V = \Sigma_{i=1}^m V_i$, where $V_i \in PM(D_i)$ and $\{D_i\}_{i=1}^m$ is a set of disjoint line segments in R^m whose direction vectors form an independent set. It is easy to show that the range of $\hat{V}$ equals the sum of the ranges of the $\hat{V}_i$, and consequently that $\Sigma_{i=1}^m \|V_i\|_{PM} \leqq 12\|V\|_{PM}$. If $D = \bigcup D_i$ is considered as a subset of T^m instead of R^m, and if $D \subset \{(x_i) \in T^m : |x_i| < \varepsilon_i\}$, then by standard arguments $\Sigma_{i=1}^m \| V_i \|_{PM} \leqq (12\Pi_{i=1}^m (1+2\varepsilon_i))\| V\|_{PM}$. The set E defined above is the union of parts F_n, F_{nkj} contained in disjoint line segments whose direction vectors form an independent set; and E lies inside a set of the form $\{(x_i) \in T^\infty : |x_i| < \varepsilon_i\}$ such that $\Pi_{i=1}^\infty (1+2\varepsilon_i) < \infty$. It follows that for every $U \in PM(E)$, the restrictions U_n, U_{nkj} of U to F_n, F_{nkj}, respectively, are well defined, and that there is a constant Q independent of U such that

$$\sum_n \|U_n\|_{PM} + \sum_{n,k,j} \|U_{nkj}\|_{PM} \leqq Q\|U\|_{PM}.$$

Therefore, in order to show that $U \in N(E)$, it suffices to show that each of the restrictions is in $N(E)$. It is an easy exercise to show that, in fact, $U_{nkj} \in M(F_{nkj})_1$, $U_n \in M(F_n \cup H_n)_1$ for $n > 0$, and $U_0 \in M(E)_2$. Thus in particular, E obeys synthesis and the order of $M(E)$ is at most 2.

Let S_0 denote the copy of S that lives on F_0, defined by $\hat{S}_0(u_0, u_1, \cdots) = \hat{S}(u_0)$. We shall prove that $S_0 \notin M(E)_1$, and hence that the order of $M(E)$ is at least 2, by showing that if $\{\mu^{(i)}\}$ is a sequence in $M(E)$ that converges to some $U_0 \in PM(F_0)$, then $U_0 \in N(F_0)$. Let $b = \sup_i \|\mu^{(i)}\|_{PM}$. Fix $m \geqq 1$. For every n, the restriction of $\mu^{(i)}$ to $F_n \cup H_n$ converges weak* to zero. Therefore we may suppose without loss of generality that $\mu^{(i)}$ vanishes on $F_n \cup H_n$ for $1 \leqq n < m$ and for all i. Let $\nu_k^{(i)}$ and $\rho^{(i)}$ be the restrictions of $\mu^{(i)}$ to $\bigcup_{n=m}^\infty \bigcup_{j=1}^\infty F_{nkj}$ and $F_0 \cup \bigcup_{n=m}^\infty F_n$, respectively. We may suppose that $\rho^{(i)} \to \rho_0 + \rho_1$ weak*, where $\rho_0 \in N(F_0)$ and $\rho_1 \in PM(\bigcup_{n=m}^\infty F_n)$. Let ν_k be the weak* limit of $\nu_k^{(i)}$. Then $\Sigma_{k=1}^{2m} \nu_k = U - \rho_0 - \rho_1$, and the restriction of ν_k to F_0 is $(U - \rho_0)_{mk}$. Therefore

$$\sum_{k=1}^{2m} \|(U_0 - \rho_0)_{mk}\|_{PM} \leqq Q \sum_k \|\nu_k\|_{PM} \leqq Q \sum_k \limsup_i \|\nu_k^{(i)}\|_{PM} \leqq Q^2 \sup_i \|\sum_k \nu_k^{(i)}\|_{PM}$$

$$\leqq Q^3 \sup_i \|\mu^{(i)}\|_{PM} \leqq Q^3 b.$$

Since m is arbitrary, it follows that $U_0 - \rho_0$ is a measure and hence $U_0 \in N(F_0)$. The theorem is proved for the case $s = 2$.

We shall now describe the general induction step that proves the theorem for the case of an arbitrary countable ordinal number $s > 2$. Our inductive hypothesis is as follows. For each t, $1 < t < s$, there is a set

$$E = E(t) = F_0 \cup \bigcup_{n=1}^{\infty} (F_n \cup H_n)$$

such that the order of $M(E)$ is t. Each F_n is a copy of F and carries a copy S_n of S. Let $M(E)_0 = M(E)$. The pseudomeasure S_0 is not in $\bigcup_{u<t} M(E)_u$, and furthermore $S_n \not\in \bigcup_{u<t_0} M(F_n \cup H_n)_u$ if $t_0 < t$; but $S_n \in \bigcup_{u<t} M(F_n \cup H_n)_u$ for each $n \geqq 1$, so that $S_0 \in M(E)_t$. The sets F_n and the sets F_{nkj} that make up the H_n lie on disjoint line segments whose direction vectors are distinct and make up an orthogonal family. Thus if $U \in PM(E)$, then U is the sum of its restrictions to these sets, and the sum of the norms of these restrictions is bounded by $Q\|U\|_{PM}$.

Let us select a sequence of such sets,

$$E_p = F_0 \cup \bigcup_{n=1}^{\infty} (F_n^{(p)} \cup H_n^{(p)}) \qquad (p = 1, 2, \cdots).$$

If s has a predecessor, let all the sets E_p be the same, namely $E(s-1)$ as described above. If s is a limit ordinal, let $\{t(p)\}$ be an enumeration of the ordinals less than s, and let E_p be $E(t(p))$, requiring that the same countably infinite set of direction vectors be used in the construction of each E_p.

We shall modify the sets E_p and then put them together to form a set $\bar{E}$, obeying synthesis, such that the order of $M(\bar{E})$ is s. Let $\{f_p\}_{p=1}^{\infty} \cup \{e_{pq} : 1 \leqq p < \infty, 1 \leqq q \leqq 2^p\}$ be a set of unit vectors whose union with the set of unit vectors used in the construction of the sets E_p is an orthogonal set. Let

$$\bar{F}_p = \{x(e_0 + 2^{-p} f_p) : x \in F\},$$

$$\bar{F}_{pnq} = \{y + x(2^{-p} f_p + 2^{-p-n-q} e_{pq}) : y \in F_n^{(p)} \cup H_n^{(p)},\ x = \pi y, \text{ and } x \in I_{pq}\},$$

where πy is the projection of y on e_0;

$$\bar{H}_p = \bigcup_{n,q} \bar{F}_{pnq},\quad \bar{E} = F_0 \cup \bigcup_{n=1}^{\infty} (\bar{F}_p \cup \bar{H}_p).$$

It is an easy exercise to show that $\bar{E}$ obeys synthesis and that the order of $M(\bar{E})$ is at most s. It remains to show that $S_0 \not\in M(\bar{E})_s$. To do this, it suffices to show that if $s_0 < s$ and $\{\mu^{(i)}\}$ is a sequence in $\bigcup_{t<s_0} M(\bar{E})_t$ that converges weak* to $U_0 \in PM(F_0)$, then $U_0 \in N(F_0)$. The argument proceeds like the one for the case $s = 2$.

The theorem is proved for T^∞. Analogous constructions may be carried out in other groups. For example, it may be done in the complete direct sum

$G = \Pi_{k=1}^{\infty} Z_k$, where Z_k is the finite discrete group $\{0, 1, \cdots, k-1\}$. In fact, the sets E may be constructed to be subsets of the set

$$Y = \{(x_j) \in G : x_j = 0 \text{ or } 1\}.$$

Let X be the subset of T,

$$X = \left\{ \sum_{j=1}^{\infty} (x_j / j!) : x_j = 0 \text{ or } 1 \right\}.$$

Let $A(Y)$ denote the algebra of restrictions to Y of the Fourier transforms on G. Let $A(X)$ be defined analogously. By Theorem 1.11 of [7], $A(Y)$ is isomorphic to $A(X)$. It follows that the theorem is true as stated, with $E \subset T$.

We obtained these constructions as a byproduct of efforts to solve these two open questions: (1) Is the union of two sets of synthesis also a set of synthesis? (2) Is every set of synthesis a Ditkin set? For a long list of related open problems, see [3, p. 222–223].

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